ANALYSIS OF A BRUSHLESS DC MOTOR USED AS A PASSIVE LOAD

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SUMMARY

This analysis is aimed at understanding the effect of using a three-phase brushless DC motor as a "passive load" by connecting its three winding wires to ground via three external resistors. This is sometimes termed dynamic braking, and can also be thought of as using the motor as a generator. In this configuration, there is no commutation being performed. Current flows through all three windings simultaneously.

There are two common methods of connecting the three windings (or phases) of a BLDC motor: (1) a Y-wound configuration; and (2) a Delta-wound (Δ-wound) configuration. Since the equations governing these two winding styles are different, the two configurations are analyzed separately. Interestingly, the two analyses show that the governing equations relating the operating parameters (motor position, speed, torque, and power dissipation), external resistor value, and motor properties (phase-to-phase resistance and motor constant) are independent of the winding configuration. Table 1 summarizes the bottom-line equations:

Average Load Torque, $\hat{T}_{Load}$	$\hat{T}_{Load} = \dfrac{\pi^2 \hat{k}_m^2 \dot{\theta}}{9\left(R_{p-p} + 2R_L\right)}$
External Load Resistor Value, R_L	$R_L = \dfrac{\pi^2 \hat{k}_m^2 \dot{\theta}}{18\hat{T}_{Load}} - \dfrac{R_{p-p}}{2}$
Average Power Dissipation in Motor (as heat), $\hat{P}_{motor}$	$\hat{P}_{motor} = \dfrac{R_{p-p}\pi^2 \hat{k}_m^2 \dot{\theta}^2}{36\left(\dfrac{R_{p-p}}{2} + R_L\right)^2}$
Average Power Dissipation in *EACH* Load Resistor, $\hat{P}_L$	$\hat{P}_L = \dfrac{R_L\pi^2 \hat{k}_m^2 \dot{\theta}^2}{54\left(\dfrac{R_{p-p}}{2} + R_L\right)^2}$

Table 1. Summarized equations

Symbol	Units	Description
$\varepsilon_A, \varepsilon_B, \varepsilon_C$	V	Individual winding back-EMF's (windings A, B, C)
R_L	Ω	External load resistance
R_W	Ω	Individual winding resistance (relation to R_{p-p} depends on Δ or Y)
R_{p-p}	Ω	Motor phase-to-phase resistance (measured across any two motor wires)
θ	Ω	Rotor position
$\dot{\theta}$	rads/sec	Motor speed
k_m	V-sec/rad = N-m/Amp	Individual winding constant (relation to $\hat{k}_m$ depends on Δ or Y)
$\hat{k}_m$	V-sec/rad = N-m/Amp	Average Motor Constant (=voltage constant; =torque constant)

Table 2. List of symbols

Y-WOUND MOTOR ANALYSIS

Figure 1 shows a schematic representation of the motor-load resistor configuration for a Y-wound motor. In the following analysis, we will neglect the effects of the winding inductances.

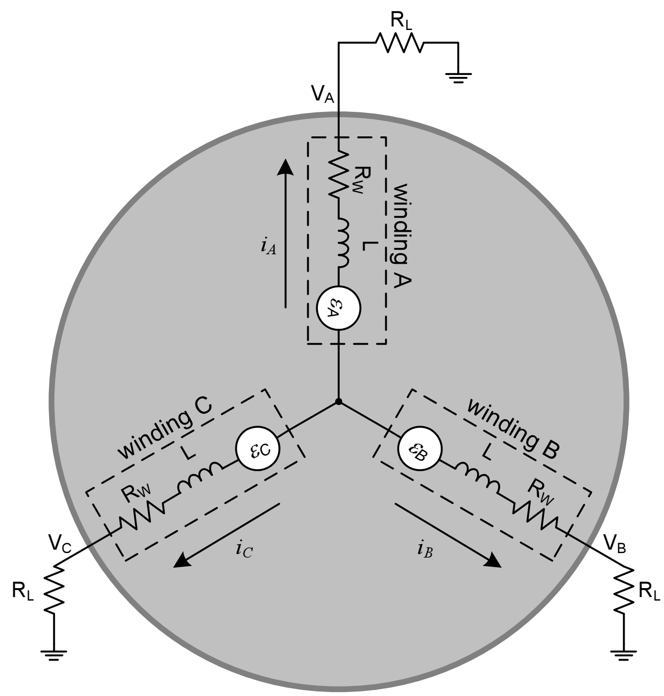

Figure 1. Diagram of 3-phase Y-wound Brushless DC Motor with external load resistors

The individual winding back EMF's are proportional to motor speed and we assume they vary sinusoidally with rotor position:

$$\varepsilon_A = k_m \dot{\theta} \sin(\theta) \tag{1.1}$$

$$\varepsilon_B = k_m \dot{\theta} \sin\left(\theta + \frac{2\pi}{3}\right) \tag{1.2}$$

$$\varepsilon_C = k_m \dot{\theta} \sin\left(\theta + \frac{4\pi}{3}\right) , \tag{1.3}$$

where θ denotes the rotor position and k_m represents the winding constant (i.e. voltage constant) for each winding. Note that this is not equal to the average voltage constant, $\hat{k}_m$, that is commonly given in motor datasheets. For a Y-wound motor, these two quantities are related by (see Appendix A):

$$k_m = \frac{\pi}{3\sqrt{3}} \hat{k}_m \approx 0.6046 \hat{k}_m . \tag{1.4}$$

One thing to note is the fact that the sum of these three back-EMF's is always zero:

$$\begin{aligned}\varepsilon_A+\varepsilon_C+\varepsilon_C &= k_m\dot{\theta}\left[\sin(\theta)+\sin\left(\theta+\frac{2\pi}{3}\right)+\sin\left(\theta+\frac{4\pi}{3}\right)\right] \\ &= k_m\dot{\theta}\begin{bmatrix}\sin(\theta)+\sin(\theta)\cos\left(\frac{2\pi}{3}\right)+\sin\left(\frac{2\pi}{3}\right)\cos(\theta)+ \\ \sin(\theta)\cos\left(\frac{4\pi}{3}\right)+\sin\left(\frac{4\pi}{3}\right)\cos(\theta)\end{bmatrix} \\ &= k_m\dot{\theta}\left\{\left[1+\cos\left(\frac{2\pi}{3}\right)+\cos\left(\frac{4\pi}{3}\right)\right]\sin(\theta)+\left[\sin\left(\frac{2\pi}{3}\right)+\sin\left(\frac{4\pi}{3}\right)\right]\cos(\theta)\right\} \\ &= k_m\dot{\theta}\left\{\left[1+(-0.5)+(-0.5)\right]\sin(\theta)+\left[(0.866)+(-0.866)\right]\cos(\theta)\right\} \\ &= 0\end{aligned} \tag{1.5}$$

Conservation of current gives:

$$i_A+i_B+i_C=0 \;\Rightarrow\; i_C=-\left(i_A+i_B\right) \tag{1.6}$$

Labeling the voltage at the center tap as V_T, we can write KVL for each winding leg:

$$\left(R_W+R_L\right)i_A=\varepsilon_A-V_T \tag{1.7}$$

$$\left(R_W+R_L\right)i_B=\varepsilon_B-V_T \tag{1.8}$$

$$\left(R_W+R_L\right)i_C=\varepsilon_C-V_T \quad , \tag{1.9}$$

where, as indicated in figure 1, R_W denotes the resistance of each winding (equal to half the phase-to-phase resistance, R_{p-p}, listed in most datasheets) and R_L denotes the resistance of each of the three external load resistors. Now solving (1.7) for V_T:

$$V_T=\varepsilon_A-\left(R_W+R_L\right)i_A \tag{1.10}$$

Plugging (1.10) into (1.8):

$$-\left(R_W+R_L\right)i_A+\left(R_W+R_L\right)i_B=\varepsilon_B-\varepsilon_A \tag{1.11}$$

Plugging (1.10) and (1.6) into (1.9):

$$-2\left(R_W+R_L\right)i_A-\left(R_W+R_L\right)i_B=\varepsilon_C-\varepsilon_A \tag{1.12}$$

Adding (1.11) and (1.12):

$$\begin{aligned}&-3\left(R_W+R_L\right)i_A=\varepsilon_C+\varepsilon_B-2\varepsilon_A \\ &\Rightarrow\; i_A=\frac{2\varepsilon_A-\varepsilon_B-\varepsilon_C}{3\left(R_W+R_L\right)}=\frac{3\varepsilon_A-\left(\varepsilon_A+\varepsilon_B+\varepsilon_C\right)}{3\left(R_W+R_L\right)}\end{aligned} \tag{1.13}$$

Now making use of (1.5), we have:

$$i_A=\frac{\varepsilon_A}{R_W+R_L}=\frac{k_m\dot{\theta}}{R_W+R_L}\sin(\theta) \tag{1.14}$$

The other two winding currents are similarly derived to be:

$$i_B=\frac{\varepsilon_B}{R_W+R_L}=\frac{k_m\dot{\theta}}{R_W+R_L}\sin\left(\theta+\frac{2\pi}{3}\right) \tag{1.15}$$

and

$$i_C=\frac{\varepsilon_C}{R_W+R_L}=\frac{k_m\dot{\theta}}{R_W+R_L}\sin\left(\theta+\frac{4\pi}{3}\right) \tag{1.16}$$

Plugging (1.13) into (1.10), we find that the center tap voltage, V_T, is equal to zero. This is sometimes termed a *virtual ground*. (If one had assumed the virtual ground at the start of the analysis, the above formulas for winding currents can be derived by inspection alone).

It is a good time to notice that the only difference between the three winding currents is a phase shift. They all have the same amplitude, but are 120 degrees apart in phase.

Continuing with the study of winding A, the torque produced by i_A is proportional to i_A and we assume it varies sinusoidally with rotor position (with the same phase as the back-EMF):

$$T_A = k_m i_A \sin(\theta) \tag{1.17}$$

Plugging (1.14) into this, we obtain an expression for the torque produced solely by i_A (as a function of rotor position):

$$T_A(\theta) = \frac{k_m^2 \dot{\theta}}{R_W + R_L} \sin^2(\theta) \tag{1.18}$$

The average of this torque over one rotor revolution is given by:

$$\hat{T}_A = \frac{1}{2\pi} \int_0^{2\pi} T_A d\theta = \frac{1}{2\pi} \frac{k_m^2 \dot{\theta}}{R_W + R_L} \int_0^{2\pi} \sin^2(\theta) d\theta = \frac{1}{2\pi} \frac{k_m^2 \dot{\theta}}{R_W + R_L} \pi \tag{1.19}$$

Simplifying:

$$\hat{T}_A = \frac{k_m^2 \dot{\theta}}{2(R_W + R_L)} \tag{1.20}$$

Again, this represents the average load torque due to the current flowing through winding A only. Windings B and C contribute the same amount of torque. The total average load torque produced by all three winding currents is simply three times this value:

$$\hat{T}_{Load} = \hat{T}_A + \hat{T}_B + \hat{T}_C = 3\hat{T}_A = \frac{3k_m^2 \dot{\theta}}{2(R_W + R_L)} \tag{1.21}$$

Substituting for k_m and R_W in terms of $\hat{k}_m$ and R_{p-p}:

$$\hat{T}_{Load} = \frac{\pi^2 \hat{k}_m^2 \dot{\theta}}{9(R_{p-p} + 2R_L)} \tag{1.22}$$

This equation expresses the average load torque produced by the load motor as a function of the motor's properties (i.e. winding resistance and voltage constant), motor speed, and external load resistance. Note that the actual load "felt" at the motor shaft will be slightly higher due to viscous and other friction losses within the load motor.

Equation (1.22) can be solved for R_L:

$$R_L = \frac{\pi^2 \hat{k}_m^2 \dot{\theta}}{18\hat{T}_{Load}} - \frac{R_{p-p}}{2} \tag{1.23}$$

If a particular load is required at a specific motor speed, equation (1.23) can be used to determine the value of the external load resistors.

To determine the power dissipation (as heat) within the load motor and the external resistors, we use the familiar relation $P = i^2 R$. The average heat dissipation in the load motor is given by:

$$\begin{aligned}
\hat{P}_{motor} &= \frac{3}{2\pi}\int_0^{2\pi} P_A d\theta = \frac{3R_W}{2\pi}\int_0^{2\pi} i_A^2 d\theta \\
&= \frac{3R_W}{2\pi}\frac{k_m^2\dot{\theta}^2}{\left(R_W + R_L\right)^2}\int_0^{2\pi}\sin^2\left(\theta\right)d\theta \\
&= \frac{3R_W k_m^2\dot{\theta}^2}{2\left(R_W + R_L\right)^2}
\end{aligned} \tag{1.24}$$

Again in terms of R_{p-p} and $\hat{k}_m$:

$$\hat{P}_{motor} = \frac{R_{p-p}\pi^2\hat{k}_m^2\dot{\theta}^2}{36\left(\frac{R_{p-p}}{2} + R_L\right)^2} \tag{1.25}$$

Similarly, the average power dissipation in ***EACH*** external load resistor is:

$$\begin{aligned}
\hat{P}_L &= \frac{1}{2\pi}\int_0^{2\pi} P_L d\theta = \frac{1}{2\pi}\int_0^{2\pi} i_A^2 R_L d\theta \\
&= \frac{R_L}{2\pi}\frac{k_m^2\dot{\theta}^2}{\left(R_W + R_L\right)^2}\int_0^{2\pi}\sin^2\left(\theta\right)d\theta \\
&= \frac{R_L k_m^2\dot{\theta}^2}{2\left(R_W + R_L\right)^2}
\end{aligned} \tag{1.26}$$

And once again in terms of R_{p-p} and $\hat{k}_m$:

$$\hat{P}_L = \frac{R_L\pi^2\hat{k}_m^2\dot{\theta}^2}{54\left(\frac{R_{p-p}}{2} + R_L\right)^2} \tag{1.27}$$

Of course there are three external load resistors, so the total average power dissipation in the external resistors is simply three times the above quantity.

DELTA-WOUND MOTOR ANALYSIS

Figure 2 shows a schematic representation of the motor-load resistor configuration for a Δ-wound motor. In the following analysis, we will neglect the effects of the winding inductances.

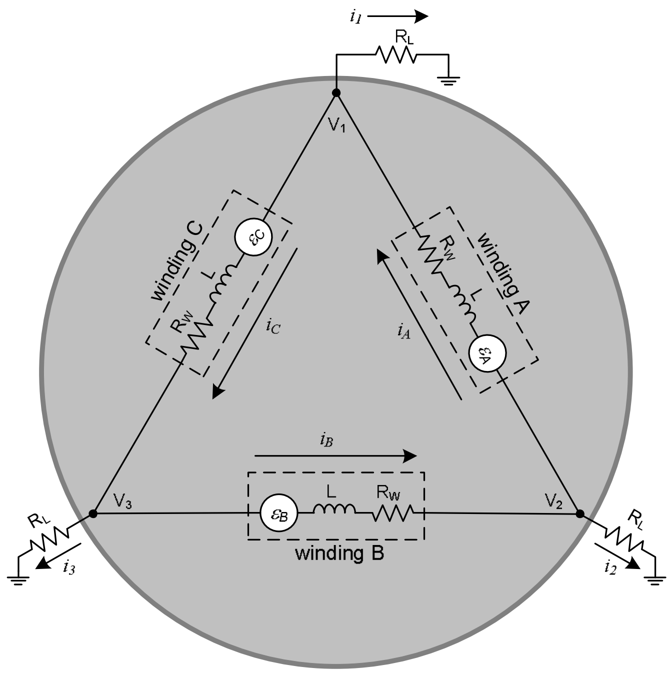

Figure 2. Diagram of 3-phase Δ-wound Brushless DC Motor with external load resistors

The individual winding back EMF's are proportional to motor speed and we assume they vary sinusoidally with rotor position:

$$\varepsilon_A = k_m \dot{\theta} \sin(\theta) \tag{2.1}$$

$$\varepsilon_B = k_m \dot{\theta} \sin\left(\theta + \frac{2\pi}{3}\right) \tag{2.2}$$

$$\varepsilon_C = k_m \dot{\theta} \sin\left(\theta + \frac{4\pi}{3}\right) , \tag{2.3}$$

where θ denotes the rotor position and k_m represents the winding constant (i.e. voltage constant) for each winding. Note that this is not equal to the average voltage constant, $\hat{k}_m$, that is commonly given in motor datasheets. For a Δ-wound motor, these two quantities are related by:

$$k_m = \frac{\pi}{3} \hat{k}_m \approx 1.047 \hat{k}_m . \tag{2.4}$$

Performing KVL on each of the three winding legs:

$$V_2 - V_1 = -\varepsilon_A + R_W i_A \tag{2.5}$$

$$V_3 - V_2 = -\varepsilon_B + R_W i_B \tag{2.6}$$

$$V_1 - V_3 = -\varepsilon_C + R_W i_C \quad , \tag{2.7}$$

where, as indicated in figure 2, R_W denotes the resistance of each winding (equal to 1.5 times the phase-to-phase resistance, R_{p-p}, listed in most datasheets).

From Ohm's law at each vertex of the delta:

$$V_1 = i_1 R_L \tag{2.8}$$

$$V_2 = i_2 R_L \tag{2.9}$$

$$V_3 = i_3 R_L \tag{2.10}$$

Conservation of current at each vertex yields:

$$i_1 = i_A - i_C \tag{2.11}$$

$$i_2 = i_B - i_A \tag{2.12}$$

$$i_3 = i_C - i_B \tag{2.13}$$

Plugging (2.11) through (2.13) into (2.8) through (2.10):

$$V_1 = \left(i_A - i_C\right) R_L \tag{2.14}$$

$$V_2 = \left(i_B - i_A\right) R_L \tag{2.15}$$

$$V_3 = \left(i_C - i_B\right) R_L \tag{2.16}$$

And plugging (2.14) through (2.16) into (2.5) through (2.7):

$$-\left(2R_L + R_W\right) i_A + R_L i_B + R_L i_C = -\varepsilon_A \tag{2.17}$$

$$R_L i_A - \left(2R_L + R_W\right) i_B + R_L i_C = -\varepsilon_B \tag{2.18}$$

$$R_L i_A + R_L i_B - \left(2R_L + R_W\right) i_C = -\varepsilon_C \tag{2.19}$$

This is a simple linear algebraic system of three equations which can be written in matrix form as follows:

$$\begin{bmatrix} -\left(2R_L + R_W\right) & R_L & R_L \\ R_L & -\left(2R_L + R_W\right) & R_L \\ R_L & R_L & -\left(2R_L + R_W\right) \end{bmatrix} \begin{bmatrix} i_A \\ i_B \\ i_C \end{bmatrix} = \begin{bmatrix} -\varepsilon_A \\ -\varepsilon_B \\ -\varepsilon_C \end{bmatrix} \tag{2.20}$$

Solving this system for the winding currents, we obtain:

$$i_A = \frac{R_L\left(\varepsilon_A + \varepsilon_B + \varepsilon_C\right) + R_W \varepsilon_A}{R_W\left(3R_L + R_W\right)} = \frac{\varepsilon_A}{3R_L + R_W} = \frac{k_m \dot{\theta}}{3R_L + R_W} \sin\left(\theta\right) \tag{2.21}$$

$$i_B = \frac{R_L\left(\varepsilon_A + \varepsilon_B + \varepsilon_C\right) + R_W \varepsilon_B}{R_W\left(3R_L + R_W\right)} = \frac{\varepsilon_B}{3R_L + R_W} = \frac{k_m \dot{\theta}}{3R_L + R_W} \sin\left(\theta + \frac{2\pi}{3}\right) \tag{2.22}$$

$$i_C = \frac{R_L\left(\varepsilon_A + \varepsilon_B + \varepsilon_C\right) + R_W \varepsilon_C}{R_W\left(3R_L + R_W\right)} = \frac{\varepsilon_C}{3R_L + R_W} = \frac{k_m \dot{\theta}}{3R_L + R_W} \sin\left(\theta + \frac{4\pi}{3}\right), \tag{2.23}$$

where we have used equation (1.5) to reduce the equations (the sum of the back-EMF's is also zero for a delta winding configuration).

Continuing with the study of winding A, the torque produced by i_A is proportional to i_A and we assume it varies sinusoidally with rotor position (with the same phase as the back-EMF):

$$T_A = k_m i_A \sin\left(\theta\right) \tag{2.24}$$

Plugging (2.21) into this, we obtain an expression for the torque produced solely by i_A (as a function of rotor position):

$$T_A(\theta) = \frac{k_m^2 \dot{\theta}}{3R_L + R_W} \sin^2(\theta) \tag{2.25}$$

The average of this torque over one rotor revolution is give by:

$$\hat{T}_A = \frac{1}{2\pi} \int_0^{2\pi} T_A d\theta = \frac{1}{2\pi} \frac{k_m^2 \dot{\theta}}{3R_L + R_W} \int_0^{2\pi} \sin^2(\theta) d\theta = \frac{1}{2\pi} \frac{k_m^2 \dot{\theta}}{3R_L + R_W} \pi \tag{2.26}$$

Simplifying, we obtain:

$$\hat{T}_A = \frac{k_m^2 \dot{\theta}}{2(3R_L + R_W)} \tag{2.27}$$

Again, this represents the average load torque due to the current flowing through winding A only. Windings B and C contribute the same amount of torque. The total average load torque produced by all three winding currents is simply three times this value:

$$\hat{T}_{Load} = \hat{T}_A + \hat{T}_B + \hat{T}_C = \frac{3k_m^2 \dot{\theta}}{2(3R_L + R_W)} \tag{2.28}$$

Substituting for k_m and R_W in terms of $\hat{k}_m$ and R_{p-p}:

$$\hat{T}_{Load} = \frac{\pi^2 \hat{k}_m^2 \dot{\theta}}{9(R_{p-p} + 2R_L)} \tag{2.29}$$

Note that this is exactly the same relation that we found for a Y-wound motor in (1.22). Solving for R_L will therefore yield the same equation as (1.23).

The average heat dissipation in the load motor is given by:

$$\begin{aligned} \hat{P}_{motor} &= \frac{3}{2\pi} \int_0^{2\pi} P_A d\theta = \frac{3R_W}{2\pi} \int_0^{2\pi} i_A^2 d\theta \\ &= \frac{3R_W}{2\pi} \frac{k_m^2 \dot{\theta}^2}{(3R_L + R_W)^2} \int_0^{2\pi} \sin^2(\theta) d\theta \\ &= \frac{3R_W k_m^2 \dot{\theta}^2}{2(3R_L + R_W)^2} \end{aligned} \tag{2.30}$$

Again in terms of R_{p-p} and $\hat{k}_m$:

$$\hat{P}_{motor} = \frac{R_{p-p} \pi^2 \hat{k}_m^2 \dot{\theta}^2}{36\left(\frac{R_{p-p}}{2} + R_L\right)^2} \tag{2.31}$$

Notice again that this is the same equation we obtained for a Y-wound motor in equation (1.25).

Similarly, the average power dissipation in ***EACH*** external load resistor is:

$$
\begin{aligned}
\hat{P}_L &= \frac{1}{2\pi}\int_0^{2\pi} P_L d\theta = \frac{1}{2\pi}\int_0^{2\pi} i_1^2 R_L d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left(i_A - i_C\right)^2 R_L d\theta \\
&= \frac{1}{2\pi}\frac{R_L k_m \dot{\theta}}{3R_L + R_W}\int_0^{2\pi}\left(\sin\left(\theta\right) - \sin\left(\theta + \frac{4\pi}{3}\right)\right)^2 d\theta \\
&= \frac{1}{2\pi}\frac{R_L k_m \dot{\theta}}{3R_L + R_W}\int_0^{2\pi}\left(\left[1 - \cos\left(\frac{4\pi}{3}\right)\right]\sin\left(\theta\right) - \sin\left(\frac{4\pi}{3}\right)\cos\left(\theta\right)\right)^2 d\theta \\
&= \frac{1}{2\pi}\frac{R_L k_m \dot{\theta}}{3R_L + R_W}\int_0^{2\pi}\left[\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\sin\left(\theta + \phi\right)\right]^2 d\theta \\
&= \frac{1}{2\pi}\frac{3R_L k_m \dot{\theta}}{3R_L + R_W}\int_0^{2\pi}\sin^2\left(\theta + \phi\right) d\theta \\
&= \frac{3R_L k_m \dot{\theta}}{2\left(3R_L + R_W\right)}
\end{aligned}
\tag{2.32}
$$

Once again, when putting this in terms of R_{p-p} and $\hat{k}_m$, we obtain:

$$
\hat{P}_L = \frac{R_L \pi^2 \hat{k}_m^2 \dot{\theta}^2}{54\left(\frac{R_{p-p}}{2} + R_L\right)^2}, \tag{2.33}
$$

just as for the Y-wound configuration.

APPENDIX A - MOTOR CONSTANT CONVERSION

For a Y-wound motor

The phase-to-phase voltage across windings A and B is given by:

$$
\begin{aligned}
V_{AB} &\equiv V_A - V_B = \varepsilon_A - \varepsilon_B \\
&= k_m\dot{\theta}\left[\sin(\theta) - \sin\left(\theta + \frac{2\pi}{3}\right)\right] \\
&= k_m\dot{\theta}\left[\sin(\theta) - \sin(\theta)\cos\left(\frac{2\pi}{3}\right) - \sin\left(\frac{2\pi}{3}\right)\cos(\theta)\right] \\
&= k_m\dot{\theta}\left\{\left[1-\cos\left(\frac{2\pi}{3}\right)\right]\sin(\theta) + \left[-\sin\left(\frac{2\pi}{3}\right)\right]\cos(\theta)\right\} \\
&= k_m\dot{\theta}\sqrt{\left[1-\cos\left(\frac{2\pi}{3}\right)\right]^2 + \left[-\sin\left(\frac{2\pi}{3}\right)\right]^2}\sin(\theta+\phi)\ ;\ \phi = \tan^{-1}\left(\frac{-\sin\left(\frac{2\pi}{3}\right)}{1-\cos\left(\frac{2\pi}{3}\right)}\right)
\end{aligned}
$$

Simplifying:

$$V_{AB} = k_m\dot{\theta}\sqrt{3}\sin(\theta+\phi)$$

Differentiating:

$$\frac{dV_{AB}}{d\theta} = k_m\dot{\theta}\sqrt{3}\cos(\theta+\phi)$$

The min/max occurs when:

$$\frac{dV_{AB}}{d\theta} = 0 \ \Rightarrow\ \cos(\theta_m+\phi) = 0$$

$$\Rightarrow\ \theta_{m,n} + \phi = \frac{\pi}{2} + n\pi \quad ; \quad n = 0, \pm1, \pm2, \ldots$$

$$\Rightarrow\ \theta_{m,n} = \frac{\pi}{2} - \phi + n\pi \quad ; \quad n = 0, \pm1, \pm2, \ldots$$

Differentiate again to check for max or min:

$$\frac{d^2V_{AB}}{d\theta^2} = -k_m\dot{\theta}\sqrt{3}\sin(\theta+\phi)$$

$$\left.\frac{d^2V_{AB}}{d\theta^2}\right|_{\theta_m} = -k_m\dot{\theta}\sqrt{3}\sin\left(\frac{\pi}{2}+n\pi\right)$$

Note that maximum corresponds to points where $\frac{d^2V_{AB}}{d\theta^2} < 0$. This is the case when n=0 (among others). Therefore, we know that V_{AB} reaches its max at $\theta_{\max} = \frac{\pi}{2} - \phi$. What we're interested in is the average value of V_{AB} throughout a 60 degree range of rotor position, centered at this maximum point:

$$\hat{V}_{AB} = \frac{3}{\pi}\int_{\theta_{\max}-\frac{\pi}{6}}^{\theta_{\max}+\frac{\pi}{6}} V_{AB}\,d\theta = \frac{3}{\pi}\int_{\theta_{\max}-\frac{\pi}{6}}^{\theta_{\max}+\frac{\pi}{6}} k_m\dot{\theta}\sqrt{3}\sin\left(\theta+\phi\right)d\theta$$

$$= \frac{3\sqrt{3}}{\pi}k_m\dot{\theta}\int_{\frac{\pi}{2}-\phi-\frac{\pi}{6}}^{\frac{\pi}{2}-\phi+\frac{\pi}{6}} \sin\left(\theta+\phi\right)d\theta$$

$$= \frac{3\sqrt{3}}{\pi}k_m\dot{\theta}\int_{\frac{\pi}{3}-\phi}^{\frac{2\pi}{3}-\phi} \sin\left(\theta+\phi\right)d\theta$$

$$= \frac{3\sqrt{3}}{\pi}k_m\dot{\theta}\left[-\cos\left(\theta+\phi\right)\right]\Big|_{\frac{\pi}{3}-\phi}^{\frac{2\pi}{3}-\phi}$$

$$= \frac{3\sqrt{3}}{\pi}k_m\dot{\theta}\left[-\cos\left(\frac{2\pi}{3}-\phi+\phi\right)+\cos\left(\frac{\pi}{3}-\phi+\phi\right)\right]$$

$$= \frac{3\sqrt{3}}{\pi}k_m\dot{\theta}\left[-\cos\left(\frac{2\pi}{3}\right)+\cos\left(\frac{\pi}{3}\right)\right]$$

$$= \frac{3\sqrt{3}}{\pi}k_m\dot{\theta}\left[0.5+0.5\right]$$

$$= \underbrace{\frac{3\sqrt{3}}{\pi}k_m}_{\hat{k}_m}\dot{\theta}$$

From this we conclude:

$$\hat{k}_m = \frac{3\sqrt{3}}{\pi}k_m$$

For a Δ-wound motor:

The phase-to-phase voltage across from V_1 to V_2 is given by:

$$
\begin{aligned}
V_{AB} &\equiv V_A - V_B = \varepsilon_A - \varepsilon_B \\
&= k_m\dot{\theta}\left[\sin(\theta) - \sin\left(\theta + \frac{2\pi}{3}\right)\right] \\
&= k_m\dot{\theta}\left[\sin(\theta) - \sin(\theta)\cos\left(\frac{2\pi}{3}\right) - \sin\left(\frac{2\pi}{3}\right)\cos(\theta)\right] \\
&= k_m\dot{\theta}\left\{\left[1 - \cos\left(\frac{2\pi}{3}\right)\right]\sin(\theta) + \left[-\sin\left(\frac{2\pi}{3}\right)\right]\cos(\theta)\right\} \\
&= k_m\dot{\theta}\sqrt{\left[1 - \cos\left(\frac{2\pi}{3}\right)\right]^2 + \left[-\sin\left(\frac{2\pi}{3}\right)\right]^2}\sin(\theta + \phi)\ ;\ \phi = \tan^{-1}\left(\frac{-\sin\left(\frac{2\pi}{3}\right)}{1 - \cos\left(\frac{2\pi}{3}\right)}\right)
\end{aligned}
$$

$$
V_{12} \equiv V_1 - V_2 = \varepsilon_A = k_m\dot{\theta}\sin(\theta)
$$

By inspection, max occurs at $\theta = \frac{\pi}{2}$. What we're interested in is the average value of V_{12} throughout a 60 degrees range of rotor position, centered at this maximum point:

$$
\begin{aligned}
\hat{V}_{12} &= \frac{3}{\pi}\int_{\frac{\pi}{2}-\frac{\pi}{6}}^{\frac{\pi}{2}+\frac{\pi}{6}} V_{12}\,d\theta = \frac{3}{\pi}\int_{\frac{\pi}{2}-\frac{\pi}{6}}^{\frac{\pi}{2}+\frac{\pi}{6}} k_m\dot{\theta}\sin(\theta)\,d\theta \\
&= \frac{3}{\pi}k_m\dot{\theta}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin(\theta)\,d\theta \\
&= \frac{3}{\pi}k_m\dot{\theta}\left[-\cos(\theta)\right]\Big|_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \\
&= \frac{3}{\pi}k_m\dot{\theta}\left[-\cos\left(\frac{2\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)\right] \\
&= \frac{3}{\pi}k_m\dot{\theta}\left[0.5 + 0.5\right] \\
&= \underbrace{\frac{3}{\pi}k_m}_{\hat{k}_m}\dot{\theta}
\end{aligned}
$$

From this we conclude:

$$
\hat{k}_m = \frac{3}{\pi}k_m
$$